Student Number:
Teacher:

St George Girls High School

Mathematics Extension 2

2022 Trial HSC Examination

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in **Section I**, use the Multiple-Choice answer sheet provided

  For questions in **Section II:**
  - Answer the questions in the booklets provided
  - Start each question in a new writing booklet
  - Show relevant mathematical reasoning and/or calculations
  - Marks may not be awarded for incomplete or poorly presented solutions, or where multiple solutions are provided

Total marks: 100

Section I – 10 marks (pages 3 – 6)

- Attempt Questions 1– 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7 –12)

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Q1-10	/10
Q11	/15
Q12	/15
Q13	/15
Q14	/15
Q15	/15
Q16	/15
TOTAL	**/100**
	%

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet provided for Questions 1 to 10.

1. Which of the following is the complex number $-2+2i$?

(A) $\sqrt{2}e^{\frac{3\pi i}{4}}$

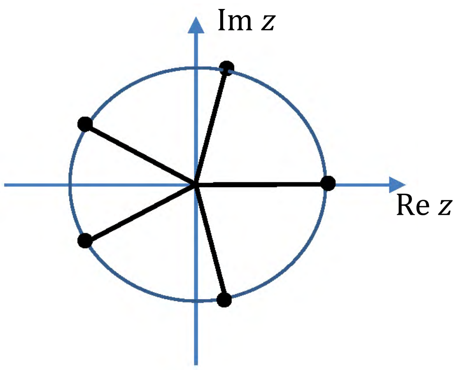
(B) $2\sqrt{2}e^{\frac{3\pi i}{4}}$

(C) $\sqrt{2}e^{-\frac{3\pi i}{4}}$

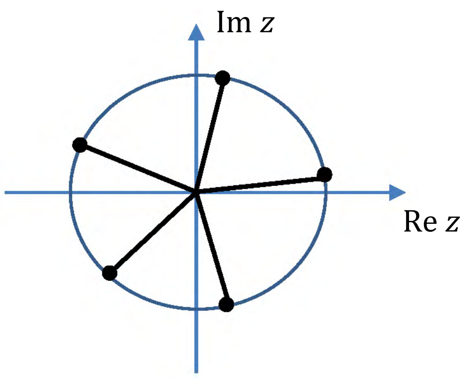
(D) $2\sqrt{2}e^{-\frac{3\pi i}{4}}$

2. Which Argand diagram shows the solutions of $z^5 - i = 0$?

(A).



(B)



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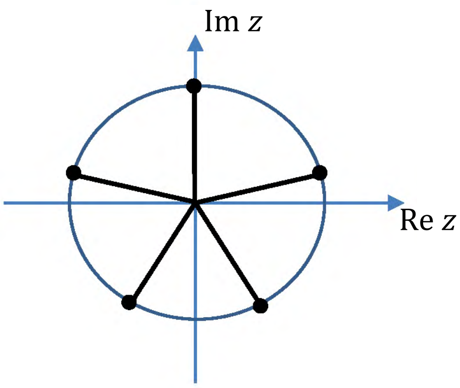
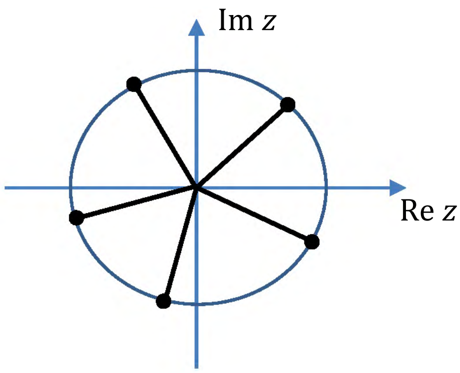


(D)



3. The vector equation of a line joining the points L $(1, 3, 5)$ and M (x, y, z) is

$$r = \begin{pmatrix}1\\3\\5\end{pmatrix} + \lambda\begin{pmatrix}2\\8\\1\end{pmatrix}.$$

Which of the following could be the coordinates of M?

(A) $(1, 5, -4)$

(B) $(3, 11, 8)$

(C) $(3, 12, 6)$

(D) $(3, 11, 6)$

4. Consider this statement about quadrilaterals:

"For all quadrilaterals, if the four sides are equal in length, then the quadrilateral is a rhombus."

Which of the following is the contrapositive of the statement?

(A) If the four sides are not equal in length, then the quadrilateral is not a rhombus.

(B) If the quadrilateral is not a rhombus, then the four sides are not equal in length.

(C) There exists a quadrilateral such that the four sides are not equal in length and the quadrilateral is not a rhombus.

(D) There exists a quadrilateral such that the quadrilateral is not a rhombus and the four sides are not equal in length.

5. Which expression is equivalent to $\int \tan^{-1}x \, dx$?

(A) $x\tan^{-1}x - \int \frac{x}{1+x^2}\, dx$

(B) $x^2\tan^{-1}x - \int \frac{x^2}{1+x^2}\, dx$

(C) $\frac{x}{2}\left(\tan^{-1}x\right)^2$

(D) $\frac{x}{2}\left(\tan^{-1}x\right)^2 - \int \frac{x}{\tan x}\, dx$

6. Consider the statement

$$\text{‘}\exists x \in \mathbb{R}, \ln x = 1 \text{ and } x > 2.\text{’}$$

Which of the following is the negation of the statement?

(A) $\exists x \in \mathbb{R}, \ln x \neq 1 \text{ and } x \leq 2$

(B) $\exists x \in \mathbb{R}, \ln x \neq 1 \text{ or } x \leq 2$

(C) $\forall x \in \mathbb{R}, \ln x \neq 1 \text{ and } x \leq 2$

(D) $\forall x \in \mathbb{R}, \ln x \neq 1 \text{ or } x \leq 2$

7. On an Argand diagram, a set of points lies on a circle of radius 3, centred at the origin. Which of the following defines this circle?

(A) $\{z \in \mathbb{C} : z\bar{z} = 3\}$

(B) $\{z \in \mathbb{C} : z^2 = 9\}$

(C) $\{z \in \mathbb{C} : (z + \bar{z})^2 - (z - \bar{z})^2 = 36\}$

(D) $\{z \in \mathbb{C} : \mathrm{Re}(z^2) + \mathrm{Im}(z^2) = 9\}$

8. The angle between the vector $x_1 \underset{\sim}{i} + y_1 \underset{\sim}{j} + z_1 \underset{\sim}{k}$ and the $x - z$ plane is:

(A) $\sin^{-1}\left(\dfrac{y_1}{\sqrt{{x_1}^2+{y_1}^2+{z_1}^2}}\right)$.

(B) $\cos^{-1}\left(\dfrac{\sqrt{{x_1}^2+{z_1}^2}}{\sqrt{{x_1}^2+{y_1}^2+{z_1}^2}}\right)$.

(C) $\cos^{-1}\left(\dfrac{y_1}{\sqrt{{x_1}^2+{y_1}^2+{z_1}^2}}\right)$.

(D) $\sin^{-1}\left(\dfrac{\sqrt{{x_1}^2+{z_1}^2}}{\sqrt{{x_1}^2+{y_1}^2+{z_1}^2}}\right)$.

9. Which expression is equal to $\int \sin^5 x \cos^2 x dx$?

(A) $-\frac{\cos^3 x}{3}+\frac{2\cos^5 x}{5}-\frac{\cos^7 x}{7}+C$

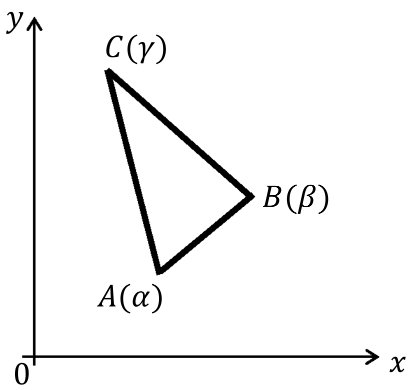
(B) $\frac{\cos^3 x}{3}-\frac{2\cos^5 x}{5}+\frac{\cos^7 x}{7}+C$

(C) $-\frac{\sin^3 x}{3}+\frac{\sin^4 x}{2}-\frac{\sin^7 x}{7}+C$

(D) $\frac{\sin^3 x}{3}-\frac{\sin^4 x}{2}+\frac{\sin^7 x}{7}+C$

10. In the diagram, ABC is a triangle such that $AC = 2AB$ and $\angle BAC = \frac{\pi}{3}$.

The vertices A, B, and C are represented by the complex numbers α, β and γ respectively.



Which of the following is the correct expression of the complex number γ ?

(A) $\gamma = (1 + i\sqrt{3})(\beta - \alpha) + \alpha$

(B) $\gamma = (1 + i\sqrt{3})(\beta + \alpha) - \alpha$

(C) $\gamma = \frac{1}{2}(1 + i\sqrt{3})(\beta - \alpha) + \alpha$

(D) $\gamma = (1 + i\sqrt{3})(\alpha - \beta) + \alpha$

END OF SECTION I

<u>Section II</u>
90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations

Question 11 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $i^{2021} + i^{2022}$. 2

(b) Consider the complex number $\alpha = 1 - i$ and $\beta = -1 - i$.
Express the following in the form $a + ib$, where a and b are real.

(i) $\alpha - \beta$ 1

(ii) β^2 1

(iii) $\alpha + \dfrac{1}{\alpha\overline{\beta}}$ 2

(c) Find $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx$ 2

(d) A, B and C are three collinear points with position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively. 2
Point B lies between A and C with $\left|\overrightarrow{BC}\right| = \frac{1}{2}\left|\overrightarrow{AB}\right|$.
Find $\underset{\sim}{c}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(e) Disprove the statement: 2

'There exists $b \in \mathbb{N}$ such that $b^2 + 9b + 20$ is a prime number.'

(f) The line l has equation $\frac{x-1}{k} = \frac{y-1}{3} = \frac{3-z}{6}$. Find the positive value of k for which 3

the angle between line l and the x −axis is 120^0.

Question 12 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Find the values of A, B, and C such that: 3

$$\frac{x^2+x+11}{(x+2)(x^2+9)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+9}.$$

(ii) Hence find $\int \frac{x^2+x+11}{(x+2)(x^2+9)}\,dx$. 2

(b) Let P and Q be points representing the complex numbers z_1 and $z_2 = -2+6i$ respectively.

(i) If ΔPOQ forms an equilateral triangle with $\arg z_2 > \arg z_1$, find the complex number represented by P, in the form $a+ib$. 3

(ii) Hence find the area of the ΔPOQ. 1

(c) What is the value of this integral correct to 2 decimal places? 3

$$\int_{0.5}^{1} \frac{x}{\sqrt{2x-x^2}}\,dx$$

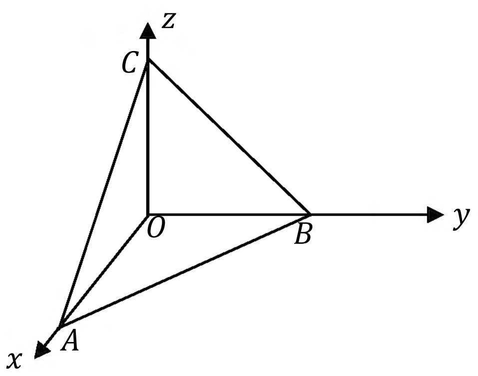
(d) Sketch the region on the Argand diagram defined by $|z-3i| > |z+2|$. 3

Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use integration by parts to find $\int e^x \sin x \, dx$. 3

(b) The diagram $OABC$ below is a right triangular pyramid in which $|\overrightarrow{OA}| = 10$, $|\overrightarrow{OB}| = 20$ and $|\overrightarrow{OC}| = 10$, and $\angle AOB = \angle AOC = \angle BOC = 90°$.

NOT DRAWN
TO SCALE



The unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are parallel to $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively.
The points R and S are the midpoints of AB and BC respectively.

(i) Determine the vectors $\overrightarrow{OR}$ and $\overrightarrow{OS}$ in terms of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$. 2

(ii) Calculate, to the nearest minute, the angle between $\overrightarrow{OR}$ and $\overrightarrow{OS}$. 2

(iii) Find the exact length of the perpendicular from R to $\overrightarrow{OS}$. 2

(c) $'mnk'$ represents a three-digit number, where m, n and k are the 1st, 2nd and 3rd digit respectively. If $m + n + k$ is divisible by three, show that mnk is divisible by three. 2

(d) If $x = 1 - 2i$ is a root of $x^3 + px^2 - x + 15$, find p if p is a real number. 2

(e) Prove by contradiction that there are no positive integers p and q, such that $4p^2 - q^2 = 25$. 2

Question 14 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A sphere has a centre at $(3, -3, 4)$ and its radius is 6 units.

A line has equation $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$.

(i) Write down the vector equation of the sphere. 1

(ii) Determine whether the line is a tangent to the sphere, clearly justifying your conclusion. 3

(b) For positive real number, x, y and z:

(i) Show that $x + \dfrac{1}{x} \geq 2$. 1

(ii) Hence show that $(x + y + z)\left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right) \geq 9$. 2

(c) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \frac{1}{3 - \cos\theta}\, d\theta$. 4

Leave your answer correct to 2 decimal places.

(d) Let n be a natural number. For each of the following statements, decide whether it is true or false. If true, give a proof. If false, give a counter example.

(i) If n is a multiple of 4 then so is n^2. 2

(ii) If n^2 is a multiple of 4 then so is n. 2

Question 15 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find α and β given that $z^3 + 6z - 4\sqrt{2}i = (z-\alpha)^2(z-\beta)$. 3

(b) Use an appropriate trigonometric substitution to show that

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.$$ 3

(c) Lines l_1 and l_2 are given below, relative to a fixed origin O.

$$l_1: \quad \underset{\sim}{r}(t) = \left(-9\underset{\sim}{i} + 10\underset{\sim}{k}\right) + \gamma\left(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\right)$$

$$l_2: \quad \underset{\sim}{r}(t) = \left(3\underset{\sim}{i} + \underset{\sim}{j} + 17\underset{\sim}{k}\right) + \mu\left(3\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k}\right)$$

where γ and μ are scalar parameters.

(i) Line l_1 meets line l_2 at the point A. What is the position vector of point A. 3

(ii) Point B has position vector $5\underset{\sim}{i} + 7\underset{\sim}{j} + 3\underset{\sim}{k}$. Does B lie on either line? 2

(d) (i) Show that $(1+i\tan\theta)^n + (1-i\tan\theta)^n = \dfrac{2\cos n\theta}{cos^n\theta}$, where $\cos\theta \neq 0, n \in \mathbb{Z}^+$. 2

(ii) By using the result in part (i), and by letting $z = i\tan\theta$, show that the roots of $(1+z)^2 + (1-z)^2 = 0$ are $z = \pm i\tan\frac{\pi}{4}$. 2

Question 16 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that $\frac{d}{dx}[(1+x)\ln(1+x)-x]=\ln(1+x)$. 2

Consider the reduction formula $I_n=\int_0^1 x^n\ln(1+x)dx$ for $n=0,1,2,\ldots$.

(ii) Show that $(n+1)I_n=2\ln 2-\frac{1}{n+1}-nI_{n-1}$, $\quad n=1,2,\ldots$. 3

(iii) Hence evaluate $4I_3$. 2

(b) Let the complex number $\alpha=\frac{1}{2}e^{i\theta}$, where θ is real.

(i) Show that the real part of the series 3

$$1+\frac{1}{2}\alpha^3+\frac{1}{4}\alpha^6+\frac{1}{8}\alpha^9+\ldots \text{ is } \frac{256-16\cos 3\theta}{257-32\cos 3\theta}.$$

(ii) Find an expression for 2

$$\frac{1}{2^4}\sin 3\theta+\frac{1}{2^8}\sin 6\theta+\frac{1}{2^{12}}\sin 9\theta+\ldots$$ in terms of $\sin 3\theta$ and $\cos 3\theta$.

(c) For d, an integer when $d>1$, show that:

(i) $\frac{1}{d^2}<\frac{1}{d(d-1)}$. 1

(ii) Noting that $\frac{1}{d^2-d}=\frac{1}{d-1}-\frac{1}{d}$, 2

show that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}<2$ as $n\longrightarrow\infty$.

END OF EXAMINATION

Mathematics Ext 2 Trial HSC 2022 Solutions & Marking Criteria

MATHEMATICS EXTENSION 2 – QUESTION Q1 – 10 (Multiple Choice)

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS

1. Let $z = -2 + 2i$

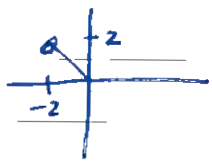
$|z| = \sqrt{(-2)^2 + 2^2}$

$= \sqrt{4+4}$

$= \sqrt{8}$

$= 2\sqrt{2}$

$\arg z = \tan^{-1}\left(\frac{-2}{2}\right)$



$= \pi - \tan^{-1} 1$

$= \pi - \frac{\pi}{4}$

$= \frac{3\pi}{4}$

$\therefore \; -2+2i = 2\sqrt{2}e^{\frac{3\pi}{4}i}$

B

2. $z^5 = 1$

Let $z = r(\cos\theta + i\sin\theta)$

$= r\,\text{cis}\,\theta$

Now $r^5(\text{cis}\,\theta)^5 = \text{cis}\,\frac{\pi}{2}$

$r^5\,\text{cis}\,5\theta = \text{cis}\,\frac{\pi}{2}$

$r^5 = 1$ $\qquad 5\theta = \frac{\pi}{2} + 2k\pi$

$\theta = \frac{\pi}{10} + \frac{2k\pi}{5}$

$\therefore z = \cos\left(\frac{\pi}{10} + \frac{2k\pi}{5}\right) + i\sin\left(\frac{\pi}{10} + \frac{2k\pi}{5}\right)$

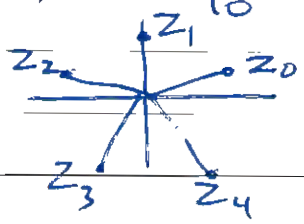
$k=0$, $\theta = \frac{\pi}{10}$, $z_0 = \text{cis}\,\frac{\pi}{10}$

$k=1$, $\theta = \frac{\pi}{2}$, $z_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$k=2$, $\theta = \frac{9\pi}{10}$, $z_2 = \text{cis}\,\frac{9\pi}{10}$

$k=-1$, $\theta = -\frac{3\pi}{10}$, $z_3 = \text{cis}\left(-\frac{3\pi}{10}\right) = \text{cis}\,\frac{17\pi}{10}$

$k=-2$, $\theta = -\frac{7\pi}{10}$, $z_4 = \text{cis}\left(-\frac{7\pi}{10}\right) = \text{cis}\,\frac{13\pi}{10}$



C

MATHEMATICS EXTENSION 2 – QUESTION Multiple choice		
SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS

3. $\vec{LM} = \begin{pmatrix} x-1 \\ y-3 \\ z-5 \end{pmatrix}$

If $\lambda = 1$, $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 8 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 11 \\ 6 \end{pmatrix}$ D

4. Original statement: $x \to y$
If the four sides are equal in length then the quadrilateral is a rhombus
Contrapositive: $\neg y \to \neg x$
If the quadrilateral is not a rhombus then the four sides are not equal in length. B

5. $\int \tan^{-1} x \, dx$ $\quad v' = 1,\ u = \tan^{-1} x$, $\quad v = x,\ u' = \frac{1}{1+x^2}$

$= uv - \int vu' \, dx$

$= x\tan^{-1} x - \int x \cdot \frac{1}{1+x^2} \, dx$ A

6. $\exists x \in R,\ \ln x = 1$ and $x > 2$
Negation
$\forall x \in R,\ \ln x \neq 1$ or $x \geqslant 2$. D

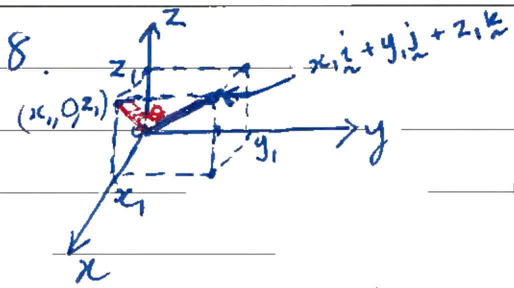
7. A. $(x+iy)(x-iy) = 3$
$x^2 + y^2 = 3$ ✗

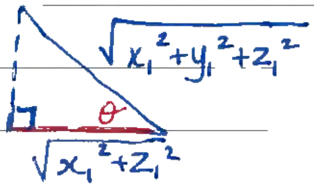
B. $(x+iy)^3 = 9$
$x^2 + 2xiy - y^2 = 9$ ✗

C. $(x+iy+x-iy)^2 - (x+iy-x+iy)^2 = 36$
$(2x)^2 - (2iy)^2 = 36$
$4x^2 - 4x - y^2 = 36$
$4(x^2+y^2) = 36 \therefore x^2+y^2 = 9$ C

MATHEMATICS EXTENSION 2 – QUESTION Multiple Choice		
SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS

8.





$$\therefore \cos\theta = \frac{\sqrt{x_1^2+z_1^2}}{\sqrt{x_1^2+y_1^2+z_1^2}}$$

$$\theta = \cos^{-1}\frac{\sqrt{x_1^2+z_1^2}}{\sqrt{x_1^2+y_1^2+z_1^2}}$$ B

9. $I = \int \sin^5 x \cos^2 x\, dx$

$= \int \sin x \cdot \sin^4 x \cos^2 x\, dx$

$= \int \sin x\,(1-\cos^2 x)^2 \cos^2 x\, dx$

Let $u = \cos x$

$\therefore du = -\sin x\, dx$

$\therefore I = \int (1-\cos^2 x)^2 \cos^2 x \sin x\, dx$

$= -\int (1-u^2)^2 u^2\, du$

$= -\int (1-2u^2+u^4)u^2\, du$

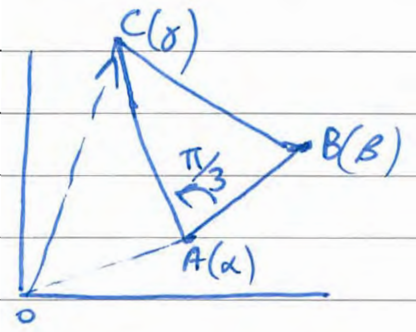
$= -\int u^2 - 2u^4 + u^6\, du$

$= -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + c$

$= -\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C$ A

MATHEMATICS EXTENSION 2 – QUESTION Multiple Choice		
SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS

10.



By rotating vector $\vec{AB}$ by $\frac{\pi}{3}$ and multiplying its magnitude by 2 we obtain $\vec{AC}$.

$$\vec{AC} = \vec{AB} \times 2 \times \text{cis}\,\frac{\pi}{3}$$

$$= (\beta-\alpha) \times 2 \times \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= (\beta-\alpha) \times 2\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

$$= (\beta-\alpha) \times (1 + i\sqrt{3})$$

but $\vec{AC} = \gamma - \alpha$

$$\gamma - \alpha = (\beta-\alpha)(1+i\sqrt{3})$$

$$\gamma = (\beta-\alpha)(1+i\sqrt{3}) + \alpha$$

Marks: A

MATHEMATICS EXTENSION 2 – QUESTION 11

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) $i^{2021} = i^{2020} \cdot i$ $= (i^2)^{1010} \cdot i$ $= (-1)^{1010} \cdot i$ $= 1 \times i$ $= i$		2 marks for complete solution and answer 1 mark for either i^{2021} or i^{2022} correct
$i^{2022} = i^{2020} \cdot i^2$ $= (i^2)^{1010} \cdot -1$ $= 1 \times -1$ $= -1$ $\therefore i^{2022} + i^{2020} = i - 1$		Question done well. 2
b) i) $\alpha - \beta = 1 - i - (-1 - i)$ $= 1 - i + 1 + i$ $= 2$		1 mark 1
ii) $\beta^2 = (-1-i)^2$ $= 1 + 2i - 1$ $= 2i$		½ mark ½ mark 1
iii) $\alpha + \dfrac{1}{\alpha\beta} = 1 - i + \dfrac{1}{(1-i)(-1+i)}$ $= 1 - i + \dfrac{1}{-1 + i + i + 1}$ $= 1 - i + \dfrac{1}{2i} \times \dfrac{i}{i}$ $= 1 - i + \dfrac{i}{-2}$ $= \dfrac{2 - 2i - i}{2}$ $= \dfrac{2 - 3i}{2}$ $= 1 - \dfrac{3}{2}i$		½ mark Expand correctly ½ mark for realising ½ mark. ½ mark 2

MATHEMATICS EXTENSION 2 – QUESTION 11

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS

c) $\int \frac{\cos(\ln x)}{x}\,dx$

Let $u = \ln x$

$\therefore du = \frac{1}{x}dx$

$= \int \cos u \, du$ — 1 mark

$= \sin u + c$

$= \sin(\ln x) + c$ — 1 mark (2)

d) If B divides AC into a ratio 2 : 1

then

$|\vec{BC}| = \frac{1}{2}|\vec{AB}|$ ---

Now

$\vec{BC} = \underset{\sim}{c} - \underset{\sim}{b}$ and $\vec{AB} = \underset{\sim}{b} - \underset{\sim}{a}$

$\therefore \underset{\sim}{c} - \underset{\sim}{b} = \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$ — 1 mark

$\underset{\sim}{c} = \frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a} + \underset{\sim}{b}$

$\underset{\sim}{c} = \frac{3}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$

or

$= \frac{1}{2}(3\underset{\sim}{b} - \underset{\sim}{a})$ — 1 mark

(2)

MATHEMATICS EXTENSION 2 – QUESTION 11

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
e) The negation statement is:		
"For all $b \in N$, $b^2+9b+20$ is not prime."		
Now $b^2+9b+20 = (b+4)(b+5)$		1 mark
But the product of two numbers is a composite number, that is, its not a prime number. ∴ Since the negation statement is true then the original statement is not true ∴ disproved.		1 mark
		2
Note: This question was not answered well. Many students used counter examples to disprove the statement. You can't use counter examples to disprove a statement that says 'there exists....' To disprove this statement successfully with counterexamples, you will have to use ALL the numbers that make the statement false, and that is impossible to do unless you have finite numbers		

MATHEMATICS EXTENSION 2 – QUESTION 11

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
f) $\frac{x-1}{k} = \frac{y-1}{3} = \frac{3-z}{6}$		This part was not well done by many students.
ℓ: $\frac{x-1}{k} = \frac{y-1}{3} = \frac{z-3}{-6}$		
$\therefore$ the direction vector of line ℓ is: $\begin{pmatrix} k \\ 3 \\ -6 \end{pmatrix}$	— $\frac{1}{2}$	mark
Now the direction vector of the x-axis is: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	—	$\frac{1}{2}$ mark.
Using the dot product,		
$\begin{pmatrix} k \\ 3 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \left\| \begin{matrix} k \\ 3 \\ -6 \end{matrix} \right\| \times \left\| \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right\| \cos 120^\circ$ $\cos 120^\circ = \dfrac{\begin{pmatrix} k \\ 3 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{\sqrt{k^2+3^2+(-6)^2} \times \sqrt{1^2}}$		1 mark
$-\frac{1}{2} = \frac{k \times 1 + 3 \times 0 + {}^-6 \times 0}{\sqrt{k^2+9+36}}$		
$-\frac{1}{2} = \frac{k}{\sqrt{k^2+45}}$ $-2k = \sqrt{k^2+45}$	— $\frac{1}{2}$	mark
$4k^2 = k^2+45$		
$3k^2 = 45$		
$k^2 = 15$		
$k = \pm\sqrt{15}$ but $k>0$		3
$\therefore$ $k = \sqrt{15}$	—	$\frac{1}{2}$ mark

MATHEMATICS EXTENSION 2 – QUESTION 12

a) i $\dfrac{x^2+x+11}{(x+2)(x^2+9)} \equiv \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+9}$

$\therefore\ x^2+x+11 \equiv A(x^2+9) + (Bx+C)(x+2)$

Coefficients of x^2: $A+B=1$

Constants: $9A+2C=11$

when $x=-2$, $4-2+11=13A$

$\therefore A=1$

$\therefore B=0$

$\therefore C=1$

3 marks Complete solution

2 marks Correctly finds at least one of A, B, or C

1 mark Significant progress to finding A, B, or C

Please state your full answer at the end; don't make the marker search through your working to find each part.

ii $\displaystyle\int \frac{x^2+x+11}{(x+2)(x^2+9)}\,dx = \int\left(\frac{1}{x+2}+\frac{1}{x^2+9}\right)dx$

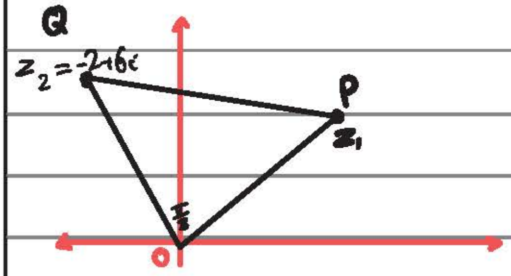
$= \ln|x+2| + \frac{1}{3}\tan^{-1}\frac{x}{3} + C$

2 marks Complete solution

1 mark Correctly finds $\ln|x+2|$ OR $\frac{1}{3}\tan^{-1}\frac{x}{3}$

Generally very well done.

b) i



$$z_1 = z_2 \times \left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$
$$= (-2+6i) \times \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$
$$= -1 + 3i + \sqrt{3}i + 3\sqrt{3}$$
$$= 3\sqrt{3} - 1 + i(3+\sqrt{3})$$

3 marks Full solution
2 marks Correct expression for z_1 (unsimplified)
or Correctly identifies $|z_1|$ AND rotation factor
1 mark Correctly identifies $|z_1|$ OR rotation factor

Best answers drew a large, clear diagram to work from. A number of students gave answers that couldn't possibly produce an equilateral triangle

ii $|z_1| = |z_2| = |-2+6i|$
$$= \sqrt{4+36}$$
$$= \sqrt{40}$$
$$= 2\sqrt{10}$$

$\therefore$ Area $= \frac{1}{2} \times 2\sqrt{10} \times 2\sqrt{10} \times \sin\frac{\pi}{3}$
$$= 20 \times \frac{\sqrt{3}}{2}$$
$$= 10\sqrt{3} \text{ units}^2$$

c)
$$\int_{\frac{1}{2}}^{1} \frac{x}{\sqrt{2x-x^2}}\,dx = -\int_{\frac{1}{2}}^{1} \frac{-x}{\sqrt{2x-x^2}}\,dx$$

$$= -\int_{\frac{1}{2}}^{1} \frac{1-x+1}{\sqrt{2x-x^2}}\,dx$$

$$= -\int_{\frac{1}{2}}^{1} \left[\frac{1-x}{\sqrt{2x-x^2}} + \frac{1}{\sqrt{2x-x^2}}\right]dx$$

$$= -\frac{1}{2}\int_{\frac{1}{2}}^{1} (2-2x)(2x-x^2)^{-\frac{1}{2}}\,dx - \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1-(x-1)^2}}\,dx$$

$$= -\frac{1}{2}\left[2(2x-x^2)^{\frac{1}{2}}\right]_{\frac{1}{2}}^{1} - \left[\sin^{-1}(x-1)\right]_{\frac{1}{2}}^{1}$$

$$= -\left[\sqrt{1} - \sqrt{\tfrac{3}{4}}\right] - \left[\sin^{-1}0 - \sin^{-1}\left(-\tfrac{1}{2}\right)\right]$$

$$= 0.389624\ldots$$

$$\doteqdot 0.39$$

3 marks Full solution

2 marks Correct integration

1 mark Correctly factorises denominator

OR
$$\int_{\frac{1}{2}}^{1} \frac{x}{\sqrt{2x-x^2}}\,dx = \int_{\frac{1}{2}}^{1} \frac{x}{\sqrt{1-(x-1)^2}}\,dx$$

let $u = x-1 \Rightarrow x = u+1$

$\frac{du}{dx} = 1$

$du = dx$

when $x = \frac{1}{2}$, $u = -\frac{1}{2}$

$x = 1$, $u = 0$

$$= \int_{-\frac{1}{2}}^{0} \frac{u+1}{\sqrt{1-u^2}}\,du$$

$$= \int_{-\frac{1}{2}}^{0} \left[\frac{u}{\sqrt{1-u^2}} + \frac{1}{\sqrt{1-u^2}}\right]du$$

$$= \int_{-\frac{1}{2}}^{0} u(1-u^2)^{-\frac{1}{2}}\,du + \int_{-\frac{1}{2}}^{0} \frac{1}{\sqrt{1-u^2}}\,du$$

$$= -\frac{1}{2}\int_{-\frac{1}{2}}^{0} -2u(1-u^2)^{-\frac{1}{2}}\,du + \left[\sin^{-1}u\right]_{-\frac{1}{2}}^{0}$$

$$= -\frac{1}{2}\left[2(1-u^2)^{\frac{1}{2}}\right]_{-\frac{1}{2}}^{0} + \left[\sin^{-1}0 - \sin^{-1}\left(-\tfrac{1}{2}\right)\right]$$

$$= -\left[1 - \sqrt{\tfrac{3}{4}}\right] + \left[0 - \sin^{-1}\left(-\tfrac{1}{2}\right)\right]$$

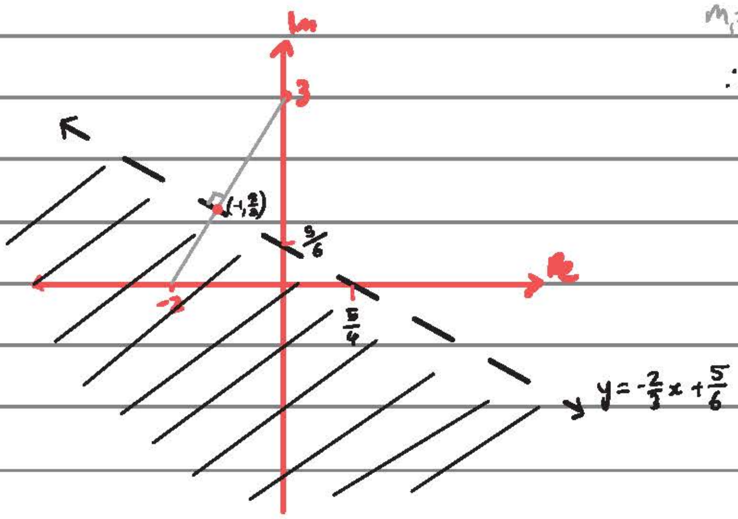
$$= 0.389624\ldots$$

$$\doteqdot 0.39$$

Please write clearly! This was a difficult question, and we're trying to give you marks for correct working, but it must be legible.

d) $|z-3i| > |z - -2|$

i.e. [the distance from $3i$] > [the distance from -2]



$m_1 = \frac{3}{2}$

$\therefore m_2 = -\frac{2}{3}$

3 marks Full solution

2 marks Correct line (algebraically or graphically)

OR Correct region, but missing labelled intercepts

1 mark Clearly identifies [distance from $3i$] and [distance from -2]

This question could also be done algebraically:

Given $|z-3i| > |z+2|$, let $z = x+iy$

$\therefore |x+iy-3i| > |x+iy+2|$

$|x+i(y-3)| > |x+2+iy|$

$\therefore \sqrt{x^2+(y-3)^2} > \sqrt{(x+2)^2+y^2}$

$x^2+y^2-6y+9 > x^2+4x+4+y^2$

$-6y+9 > 4x+4$

$6y < -4x+5$

$y < -\frac{2}{3}x+\frac{5}{6}$

Note that any algebra was only to support your answer: the most important thing was your graph.

MATHEMATICS EXTENSION 2 – QUESTION 13

a) $\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$

$u = \sin x \quad v = e^x$

$u' = \cos x \quad v' = e^x$

$= e^x \sin x - \left[e^x \cos x - \int -e^x \sin x\,dx\right]$

$u = \cos x \quad v = e^x$

$u' = -\sin x \quad v' = e^x$

$= e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$

$\therefore 2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x$

$\therefore \int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C$

3 marks Complete solution

2 marks Correctly uses integration by parts once

1 mark Clearly attempts to use integration by parts

Generally well done, but don't forget the "+C"!

b) i $S = \left(\frac{0+0}{2}, \frac{0+20}{2}, \frac{10+0}{2}\right)$

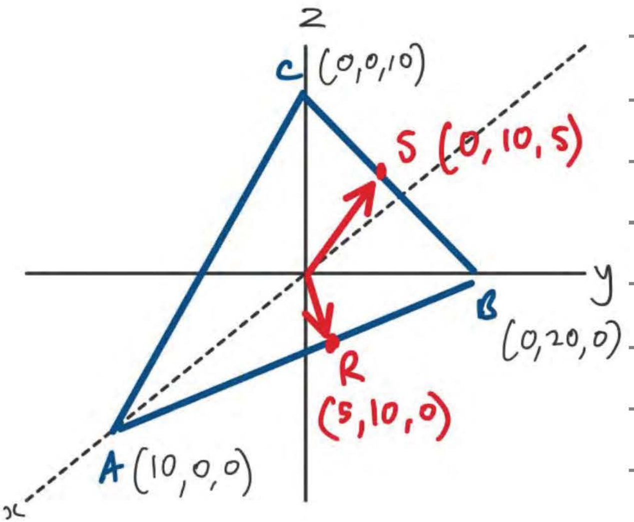
$= (0, 10, 5)$

$\therefore \vec{OS} = 10\underset{\sim}{j} + 5\underset{\sim}{k}$

$R = \left(\frac{10+0}{2}, \frac{0+20}{2}, \frac{0+0}{2}\right)$

$= (5, 10, 0)$

$\therefore \vec{OR} = 5\underset{\sim}{i} + 10\underset{\sim}{j}$



2 marks Complete solution

1 mark Correctly finds $\vec{OS}$ or $\vec{OR}$

or Clearly attempts to find the midpoint

Many students made this question much more time consuming than necessary.
Better responses included a quick diagram.

b) ii

$$\cos\theta = \frac{\begin{bmatrix}0\\10\\5\end{bmatrix}\cdot\begin{bmatrix}5\\10\\0\end{bmatrix}}{\left|\begin{bmatrix}0\\10\\5\end{bmatrix}\right|\left|\begin{bmatrix}5\\10\\0\end{bmatrix}\right|}$$

$$= \frac{100}{\sqrt{125}\times\sqrt{125}}$$

$$= \frac{4}{5}$$

$$\therefore \theta = \cos^{-1}\frac{4}{5}$$

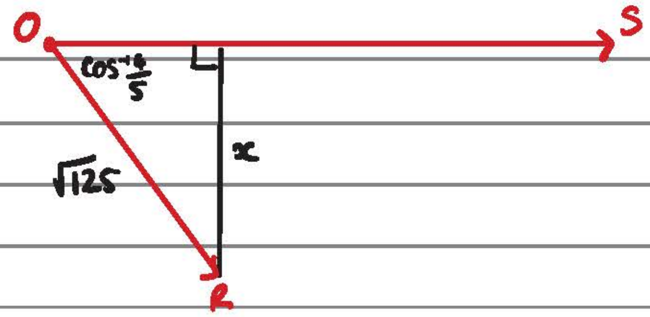
$$= 36.8698\ldots$$

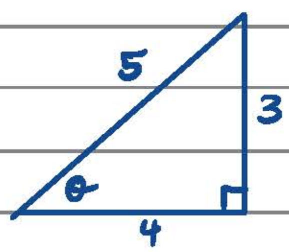
$$\doteqdot 36^\circ 52'$$

2 marks Complete solution

1 mark Correct dot product or correct $|\vec{OR}|$ or $|\vec{OS}|$

iii





$$\sin\left(\cos^{-1}\frac{4}{5}\right) = \frac{x}{\sqrt{125}}$$

$$\therefore \frac{3}{5} = \frac{x}{\sqrt{125}}$$

$$\therefore x = \sqrt{125}\times\frac{3}{5}$$

$$= 3\sqrt{5}$$

2 marks Complete solution

1 mark Correct approximate/unsimplified answer

Draw a diagram! This question can be solved easily using trigonometry.

c) Let $m+n+k=3P$ $\quad P\in\mathbb{Z}$ (since $m+n+k$ is divisible by 3)

$$\text{"mnk"} = 100m+10n+k$$
$$= 99m+9n+m+n+k$$
$$= 99m+9n+3P$$
$$= 3(33m+3n+P)$$

which is divisible by 3

2 marks Complete solution

1 mark Correctly writes "mnk" as $100m+10n+k$

d) Given $1-2i$ is a root,

$$(1-2i)^3+p(1-2i)^2-(1-2i)+15=0$$
$$-11+2i+p(-3-4i)-1+2i+15=0$$
$$p(-3-4i)=-3-4i$$
$$\therefore p=1$$

2 marks Complete solution

1 mark Correctly substitutes $x=1-2i$ as a root

e) Suppose there are positive integers p and q such that $4p^2-q^2=25$

$$\therefore (2p-q)(2p+q)=25$$

Since p and q are integers, $(2p-q)$ and $(2p+q)$ are also integers

Since the integer factors of 25 are 1, 5, and 25,

$(2p+q)$ and $(2p-q)$ are either 5 and 5, or 25 and 1.

If they are 5 and 5, then $q=0$, which is a contradiction $(q>0)$

$\therefore 2p+q=25$ and $2p-q=1$

$$\therefore 4p=26$$

$p=\frac{13}{2}$ which is a contradiction (p is an integer)

$\therefore$ the supposition is false

$\therefore$ there are no positive integers p and q such that $4p^2-q^2=25$

2 marks Complete solution

1 mark Clearly states assumption of negation in full.

MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) i) vector equation of a sphere is: $\left\lvert \underset{\sim}{r} - \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} \right\rvert = 6$		1 mark. Some students wrote the Cartesian equation of a sphere.
ii) For the line $x = 1 + 2\lambda$, $y = 5 - \lambda$, $z = 4 - \lambda$ (*)	½	This part of the question was not answered well.
The cartesian equation of a sphere is		
$(x-3)^2 + (y+3)^2 + (z-4)^2 = 36$		
sub (*)		
$((1+2\lambda)-3)^2 + ((5-\lambda)+3)^2 + (4-\lambda-4)^2 = 36$		
$(2\lambda-2)^2 + (8-\lambda)^2 + (-\lambda)^2 = 36$	1	
$4\lambda^2 - 8\lambda + 4 + 64 - 16\lambda + \lambda^2 + \lambda^2 = 36$		
$6\lambda^2 - 24\lambda + 36 = 0$	½	
Now $\Delta = b^2 - 4ac$ $= 24^2 - 4 \times 6 \times 36$ $= -288$ < 0		
$\therefore$ This quadratic has no real solutions since $\Delta < 0$ so the the line does not intersect the sphere.		1 mark
$\therefore$ the line is NOT a tangent to the sphere.		

3

MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) ii) Alternate solution		
From (*) sub in vector equation of the line		
$\left\lvert \begin{pmatrix} 1+2\lambda \\ 5-\lambda \\ 4-\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} \right\rvert = 6$		
$\left\lvert \begin{matrix} -2+2\lambda \\ 8-\lambda \\ -\lambda \end{matrix} \right\rvert = 6$	1	
Now $-2+2\lambda=6$, $2\lambda=8$, $\lambda=4$		
$8-\lambda=6$, $\lambda=2$ and $-\lambda=6$, $\lambda=-6$	$1\frac{1}{2}$	
Since λ is not consistent $\therefore$ the line is not a tangent to the sphere.	$\frac{1}{2}$	3

MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
b) i) Consider $(x-1)^2 \geq 0$ $x^2 - 2x + 1 \geq 0$ $x^2 + 1 \geq 2x$ $x + \frac{1}{x} \geq 2$ since $x > 0$ inequality sign holds. Alternate solution: Consider $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \geq 0$ $x - 2 + \frac{1}{x} \geq 0$ $x + \frac{1}{x} \geq 2$		1 mark for correct proof. 1
ii) Now $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 1 + \frac{x}{y} + \frac{x}{z} + \frac{y}{x} + 1 + \frac{y}{z} + \frac{z}{x} + \frac{z}{y} + 1$ $= 3 + \left(\frac{x}{y}+\frac{y}{x}\right) + \left(\frac{x}{z}+\frac{z}{x}\right) + \left(\frac{y}{z}+\frac{z}{y}\right)$ Let $x \to \frac{x}{y}$, and $x \to \frac{x}{z}$, $x \to \frac{y}{z}$ $= 3 + \left(x+\frac{1}{x}\right) + \left(x+\frac{1}{x}\right) + \left(x+\frac{1}{x}\right)$ $\geq 3 + 2 + 2 + 2$ from (1) ≥ 9		½ mark ½ mark ½ mark ½ mark 2

Some students are still starting their proofs with the inequality they have to prove. (Zero marks)

MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
(c) $I=\int_0^{\frac{\pi}{3}} \frac{1}{3-\cos\theta}\,d\theta$		
If $t=\tan\frac{\theta}{2}$ $\frac{dt}{d\theta}=\frac{1}{2}\sec^2\frac{\theta}{2}$ $=\frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)$ $=\frac{1}{2}(1+t^2)$ $2\,dt=(1+t^2)\,d\theta$ $d\theta=\frac{2\,dt}{1+t^2}$	½	
When $\theta=\frac{\pi}{3}$, $t=\tan\frac{\pi}{6}$ $=\frac{1}{\sqrt{3}}$ $\theta=0$, $t=\tan 0$ $=0$	1 mark	
Now $I=\int_0^{\frac{1}{\sqrt{3}}}\frac{1}{3-\left(\frac{1-t^2}{1+t^2}\right)}\cdot\frac{2\,dt}{1+t^2}$	½ mark	
$=\int_0^{\frac{1}{\sqrt{3}}}\frac{2\,dt}{\cancel{(1+t^2)}\left[\frac{3(1+t^2)-(1-t^2)}{\cancel{1+t^2}}\right]}$ $=2\int_0^{\frac{1}{\sqrt{3}}}\frac{dt}{3+3t^2-1+t^2}$ $=2\int_0^{\frac{1}{\sqrt{3}}}\frac{dt}{4t^2+2}$ $=\cancel{2}\int_0^{\frac{1}{\sqrt{3}}}\frac{dt}{\cancel{2}(2t^2+1)}$		Many students did not replace $d\theta$ with $\frac{2dt}{1+t^2}$ or did not change the limits of integration. – Care must be taken with "integration by Substitution"
$=\int_0^{\frac{1}{\sqrt{3}}}\frac{dt}{2t^2+1}$	½	
$=\frac{1}{2}\int_0^{\frac{1}{\sqrt{3}}}\frac{dt}{t^2+\frac{1}{2}}$ $=\frac{1}{2}\left[\sqrt{2}\tan^{-1}\sqrt{2}\,t\right]_0^{\frac{1}{\sqrt{3}}}$	1	mark 4
$=\frac{\sqrt{2}}{2}\left(\tan^{-1}\sqrt{2}\left(\frac{1}{\sqrt{3}}\right)-\tan^{-1}\sqrt{2}(0)\right)$		

$\doteqdot 0.484$ — ½ (in radians)

MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
d) i) Statement is true		
Let $n = 4m$, $m \in \mathbb{Z}^+$		
Now		
$n^2 = (4a)^2$		1mark
$= 16a^2$		
$= 4 \times 4a^2$		1mark
which is divisible by 4		
$\therefore$ If n is a multiple of 4 then so is n^2		2
ii) Statement is false		
Using a counter example		
Let $n^2 = 36$ as		
36 is a multiple of 4		1mark
but $\sqrt{36} = 6$ is not a multiple of 4		1mark
$\therefore$ If n^2 is a multiple of 4 then so is n is a false statement.		2
		This part was answered well overall.

MATHEMATICS EXTENSION 2 – QUESTION 15

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) For		This question was not well attempted.
$z^3+6z-4\sqrt{2}i=(z-\alpha)^2(z-\beta)$ ---(*)		
RHS of (*)		
$(z^2-2z\alpha+\alpha^2)(z-\beta)$		
$=z^3-2\alpha z^2+\alpha^2 z-\beta z^2+2\alpha\beta z-\alpha^2\beta$		
$=z^3+z^2(-2\alpha-\beta)+z(\alpha^2+2\alpha\beta)-\alpha^2\beta$		1 mark
From (*)		
$z^3+6z-4\sqrt{2}i=z^3+z^2(-2\alpha-\beta)+z(\alpha^2+2\alpha\beta)-\alpha^2\beta$		
Equating coefficients:		
z^2: $-2\alpha-\beta=0$		
$\beta=-2\alpha$ --- (1)		
z: $\alpha^2+2\alpha\beta=6$ ---- (2)		½ mark
constant: $\alpha^2\beta=4\sqrt{2}i$ -- (3)		
Now sub (1) into (2)		
$\alpha^2+2\alpha(-2\alpha)=6$		
$\alpha^2-4\alpha^2=6$		
$-3\alpha^2=6$		
$\alpha^2=-2$		
$\alpha=\pm\sqrt{2}i$		½ mark
Let $P(z)=z^3+6z-4\sqrt{2}i$		
Testing both values of α		½ mark for testing $\alpha=\pm\sqrt{2}i$ to see which satisfies the polynomial.
when $\alpha=\sqrt{2}i$, $P(\sqrt{2}i)=(\sqrt{2}i)^3+6(\sqrt{2}i)-4\sqrt{2}i$		
$=-2\sqrt{2}i+2\sqrt{2}i=0$		
$\alpha=-\sqrt{2}i$, $P(-\sqrt{2}i)=2\sqrt{2}i-6\sqrt{2}i-4\sqrt{2}i$		
$\neq 0$		
$\therefore \alpha=\sqrt{2}i$ is the double root		
Sub in (1)		
$\beta=-2\sqrt{2}i$		½ mark 3

MATHEMATICS EXTENSION 2 – QUESTION 15

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) Alternate solution From the line marked in the above solution Sub ① into ③ $\alpha^2(-2\alpha) = 4\sqrt{2}\,i$ $-2\alpha^3 = 4\sqrt{2}\,i$ $\alpha^3 = -2\sqrt{2}\,i$ $\alpha = -\sqrt{2} \times i^3$ $= \sqrt{2}\,i$		1 mark
Sub in ① $\beta = -2\sqrt{2}\,i$		½ mark
Note: Most students used the first method but did not test both values of $\alpha = \pm\sqrt{2}\,i$ If they gave the answer of $\alpha = \pm\sqrt{2}\,i$ and $\beta = \pm 2\sqrt{2}\,i$ then 2½ marks were awarded. Please review the Polynomials section in Complex numbers.		

MATHEMATICS EXTENSION 2 – QUESTION 15

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS

b) $LHS = \int \frac{x^2}{\sqrt{9-x^2}}\,dx$

Let $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$= \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}} \times 3\cos\theta\, d\theta$ — mark

$= \int \frac{9\sin^2\theta}{\sqrt{9(1-\sin^2\theta)}} \times 3\cos\theta\, d\theta$

$= \cancel{\frac{1}{3}} \int \frac{9\sin^2\theta}{\sqrt{\cos^2\theta}} \times \cancel{3}\cos\theta\, d\theta$

$= 9\int \frac{\sin^2\theta}{\cancel{\cos\theta}} \cdot \cancel{\cos\theta}\, d\theta$

$= 9\int \sin^2\theta\, d\theta$ — ½ mark

$= 9\int \frac{1}{2}(1-\cos 2\theta)\, d\theta$

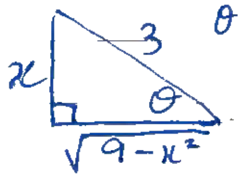
$= \frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + c$ — 1 mark

$= \frac{9}{2}\left[\theta - \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right] + c$ — ½ mark

If $x = 3\sin\theta$

$\sin\theta = \frac{x}{3}$

$\theta = \sin^{-1}\frac{x}{3}$



$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3} \times \frac{\sqrt{9-x^2}}{3}\right) + c$ — ½ mark to write integrand in terms of x

$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x\sqrt{9-x^2}}{2} + c$

$= RHS$ #

3

Note: Students were asked to use trig substitutions to integrate but many used other substitutions which made it difficult to simplify the integrand. Please review "Integration using Trig substitution."

MATHEMATICS EXTENSION 2 – QUESTION 15

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) i) For ℓ_1: $\underset{\sim}{r}(t) = (-9\underset{\sim}{i} + 10\underset{\sim}{k}) + \gamma(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$ --- (A) ℓ_2: $\underset{\sim}{r}(t) = (3\underset{\sim}{i} + \underset{\sim}{j} + 17\underset{\sim}{k}) + \mu(3\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k})$ --- (B) If ℓ_1 intersects the line ℓ_2 then $\ell_1 = \ell_2$ $-9\underset{\sim}{i} + 10\underset{\sim}{k} + \gamma(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}) = 3\underset{\sim}{i} + \underset{\sim}{j} + 17\underset{\sim}{k} + \mu(3\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k})$ $\underset{\sim}{i}(-9 + 2\gamma) + \gamma\underset{\sim}{j} + \underset{\sim}{k}(10 - \gamma) = \underset{\sim}{i}(3 + 3\mu) + \underset{\sim}{j}(1 - \mu) + \underset{\sim}{k}(17 + 5\mu)$		This part was well done. Just some minor errors when expanding & simplifying — ½ mark
Equating parts $\underset{\sim}{i}$: $-9 + 2\gamma = 3 + 3\mu$ --- (1) $\underset{\sim}{j}$: $\gamma = 1 - \mu$ --- (2) $\underset{\sim}{k}$: $10 - \gamma = 17 + 5\mu$ --- (3)		½ mark
$2 \times (2)$ $2\gamma = 2 - 2\mu$ --- (4) (1) − (4) $-9 = 1 + 5\mu$ --- (5) $\mu = -2$		½ mark
Sub $\mu = -2$ into (2) $\gamma = 1 - -2$ $\gamma = 3$		½ mark
Sub in (A) & (B) ℓ_1: $-9\underset{\sim}{i} + 10\underset{\sim}{k} + 3(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$ $= -3\underset{\sim}{i} + 3\underset{\sim}{j} + 7\underset{\sim}{k}$ also ℓ_2: $3\underset{\sim}{i} + \underset{\sim}{j} + 17\underset{\sim}{k} - 2(3\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k})$ $= -3\underset{\sim}{i} + 3\underset{\sim}{j} + 7\underset{\sim}{k}$ $\therefore$ Position vector of A is: $-3\underset{\sim}{i} + 3\underset{\sim}{j} + 7\underset{\sim}{k}$ or $\begin{pmatrix} -3 \\ 3 \\ 7 \end{pmatrix}$		1 mark 3

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c) ii) For point B $\begin{pmatrix}5\\7\\3\end{pmatrix}$ to satisfy line d_1: $\begin{pmatrix}-9\\0\\10\end{pmatrix}+\gamma\begin{pmatrix}2\\1\\-1\end{pmatrix}$ --- (*)

We need to find a scalar γ, where

$$\begin{pmatrix}5\\7\\3\end{pmatrix}=\begin{pmatrix}-9\\0\\10\end{pmatrix}+\gamma\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$$\gamma\begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}5\\7\\3\end{pmatrix}-\begin{pmatrix}-9\\0\\10\end{pmatrix}$$

$$=\begin{pmatrix}14\\7\\-7\end{pmatrix}$$

$$=7\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$\therefore \gamma=7$

$\therefore$ B lies on line d_1

OR From (*)

We need to find a consistent scalar γ, such that:

$-9+2\gamma=5$, $\gamma=7$

$0+\gamma=7$, $\gamma=7$

$10-\gamma=3$, $\gamma=7$

1 mark for showing that $\gamma=7$ for all three equations.

For Point B to satisfy $\begin{pmatrix}3\\1\\17\end{pmatrix}+\mu\begin{pmatrix}3\\-1\\5\end{pmatrix}$

$3+3\mu=5$, $\mu=\frac{2}{3}$

$1-\mu=7$, $\mu=-6$

$17+5\mu=3$, $\mu=-2\frac{4}{5}$

The scalar μ is not consistent

$\therefore$ Point B lies on d_1 and does not lie on d_2.

1 mark for showing μ is not consistent and stating that B lies on d_1 and not on d_2 from above working

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MATHEMATICS EXTENSION 2 – QUESTION 15

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
d) i) $LHS = (1+i\tan\theta)^n + (1-i\tan\theta)^n$		
$= \left(\frac{\cos\theta}{\cos\theta} + i\frac{\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos\theta}{\cos\theta} - i\frac{\sin\theta}{\cos\theta}\right)^n$		
$= \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n$		— 1 mark.
$= \frac{\cos n\theta + \cancel{i\sin n\theta} + \cos n\theta - \cancel{i\sin n\theta}}{\cos^n\theta}$		— 1 mark for using DeMoivres theorem and simplifying
$= \frac{2\cos n\theta}{\cos^n\theta}$ $= RHS$		2
Alternate solution:		
Let $z_1 = 1 + i\tan\theta$ $\quad z_2 = 1 - i\tan\theta$ $\|z_1\| = \sqrt{1+\tan^2\theta}$ $\quad \|z_2\| = \sqrt{1+\tan^2\theta}$ $= \sqrt{\sec^2\theta}$ $\quad = \sqrt{\sec^2\theta}$ $= \sec\theta$ $\quad = \sec\theta$		Many students used this method as well.
$\arg z_1 = \tan^{-1}(\tan\theta)$ $\quad \arg z_2 = \tan^{-1}(-\tan\theta)$ $= \theta$ $\quad = -\theta$		2 marks for complete work
$LHS = (1+i\tan\theta)^n + (1-i\tan\theta)^n$ $= (\sec\theta \operatorname{cis}\theta)^n + (\sec\theta \operatorname{cis}(-\theta))^n$ $= \sec^n\theta \operatorname{cis} n\theta + \sec^n\theta \operatorname{cis}(-n\theta)$ $= \sec^n\theta[\cos n\theta + i\sin n\theta + \cos(-n\theta) + \sin(-n\theta)]$ $= \sec^n\theta[\cos n\theta + \cancel{i\sin n\theta} + \cos n\theta - \cancel{i\sin n\theta}]$ $= \sec^n\theta \times 2\cos n\theta$ $= \frac{2\cos n\theta}{\cos^n\theta} = RHS$		1 mark for progressing to the proof and using De Moivres theorem 2

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d) ii) Let $z = i\tan\theta$ --- (1)		
Now given that		
$0 = (1+z)^2 + (1-z)^2$		
$0 = (1 + i\tan\theta)^2 + (1 - i\tan\theta)^2$		
$0 = \dfrac{2\cos 2\theta}{\cos^2\theta}$ from (i)		
$\therefore\ 2\cos 2\theta = 0$		½ mark.
$\cos 2\theta = 0$		
$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$		
$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$ (or $\theta = \pm\pi/4$)		½ mark
when $\theta = \frac{\pi}{4}$ sub in (1)		
$z_1 = i\tan(\pi/4)$		½ mark
when $\theta = \frac{3\pi}{4}$ sub in (1)		
$z_2 = i\tan\frac{3\pi}{4}$		
$= i\tan\left(-\frac{\pi}{4}\right)$		½ mark to show $i\tan\left(-\frac{\pi}{4}\right) = -i\tan\pi/4$
$= -i\tan\frac{\pi}{4}$		(Many students did not show this).
$\therefore$ the roots of $(1+z)^2 + (1-z)^2 = 0$ are $z = \pm i\tan\pi/4$.		2

MATHEMATICS EXTENSION 2 – QUESTION 16

let $u = 1+x$ $\quad v = \ln(1+x)$

$u' = 1$ $\quad v' = \dfrac{1}{1+x}$

a) i $\dfrac{d}{dx}\left[(1+x)\ln(1+x) - x\right] = vu' + uv' - 1$

$$= 1 \times \ln(1+x) + (1+x)\left(\frac{1}{1+x}\right) - 1$$

$$= \ln(1+x) + \frac{1+x}{1+x} - 1$$

$$= \ln(1+x) + 1 - 1$$

$$= \ln(1+x)$$

2 Marks Complete solution

1 Mark Shows significant progress

let $u = x^n$ $\quad v = (1+x)\ln(1+x) - x$

$u' = nx^{n-1}$ $\quad v' = \ln(1+x)$

ii $I_n = \int_0^1 x^n \ln(1+x)\,dx$

$$= \left[x^n\left\{(1+x)\ln(1+x) - x\right\}\right]_0^1 - \int_0^1 \left[nx^{n-1}\left\{(1+x)\ln(1+x) - x\right\}\right]dx$$

$$= 1\left\{2\ln 2 - 1\right\} - n\int_0^1 \left\{x^{n-1}\ln(1+x) + x^n\ln(1+x) - x^n\right\}dx$$

$$= 2\ln 2 - 1 - n\int_0^1 x^{n-1}\ln(1+x)\,dx - n\int_0^1 x^n \ln(1+x)\,dx + n\int_0^1 x^n\,dx$$

$$= 2\ln 2 - 1 - nI_{n-1} - nI_n + n\left[\frac{x^{n+1}}{n+1}\right]_0^1$$

$$= 2\ln 2 - 1 - nI_{n-1} - nI_n + \frac{n}{n+1}$$

$$\therefore nI_n + I_n = 2\ln 2 - nI_{n-1} - 1 + \frac{n}{n+1}$$

$$= 2\ln 2 - nI_{n-1} - \frac{n+1}{n+1} + \frac{n}{n+1}$$

$$= 2\ln 2 - nI_{n-1} - \frac{1+n-n}{n+1}$$

$$= 2\ln 2 - \frac{1}{n+1} - nI_{n-1}$$

3 marks Complete solution

2 marks Correct up to and including substitution for nI_{n-1} or nI_n

1 mark Correctly attempts to use integration by parts

A three-mark "show" question requires you to show every step; don't take shortcuts.

Better solutions used lots of paper and took everything step by step.

Also, fudging your answer so that it matches the final result cannot earn marks, and is in fact an error.

a) iii $4I_3 = 2\ln 2 - \frac{1}{3+1} - 3I_2$ (from part ii)

$= 2\ln 2 - \frac{1}{4} - \left[2\ln 2 - \frac{1}{2+1} - 2I_1\right]$

$= 2\ln 2 - \frac{1}{4} - 2\ln 2 + \frac{1}{3} + \left[2\ln 2 - \frac{1}{1+1} - I_0\right]$

$= 2\ln 2 - \frac{1}{4} - 2\ln 2 + \frac{1}{3} + 2\ln 2 - \frac{1}{2} - \int_0^1 \ln(1+x)\,dx$

$= 2\ln 2 - \frac{1}{4} + \frac{1}{3} - \frac{1}{2} - \left[(1+x)\ln(1+x) - x\right]_0^1$

$= 2\ln 2 - \frac{1}{4} + \frac{1}{3} - \frac{1}{2} - \left[(2\ln 2 - 1) - (1\ln 1 - 0)\right]$

$= 1 - \frac{1}{4} + \frac{1}{3} - \frac{1}{2}$

$= \frac{7}{12}$

2 marks Complete solution

1 mark Correct statement for $3I_2$ or I_0

b) i $\frac{T_2}{T_1} = \frac{\frac{1}{2}\alpha^3}{1}$ $\quad \frac{T_3}{T_2} = \frac{\frac{1}{4}\alpha^6}{\frac{1}{2}\alpha^3}$

$= \frac{1}{2}\alpha^3$

$\therefore$ GP with $r = \frac{1}{2}\alpha^3$

$= \frac{1}{2}\left(\frac{1}{2}e^{i\theta}\right)^3$

$= \frac{1}{16}e^{i3\theta}$

$\therefore S_\infty = \frac{a}{1-r}$

$= \frac{1}{1 - \frac{1}{16}e^{i3\theta}}$

$= \frac{16}{16 - (\cos 3\theta + i\sin 3\theta)}$

$= \frac{16}{(16 - \cos 3\theta) - i\sin 3\theta}$

$= \frac{16\left[(16 - \cos 3\theta) + i\sin 3\theta\right]}{\left[(16 - \cos 3\theta) - i\sin 3\theta\right]\left[(16 - \cos 3\theta) + i\sin 3\theta\right]}$

$= \frac{256 - 16\cos 3\theta + 16i\sin 3\theta}{(16 - \cos 3\theta)^2 - i^2\sin^2 3\theta}$

$= \frac{256 - 16\cos 3\theta + 16i\sin 3\theta}{256 - 32\cos 3\theta + \cos^2 3\theta + \sin^2 3\theta}$

$$= \frac{256 - 16\cos 3\theta + 16i\sin 3\theta}{256 - 32\cos 3\theta + 1}$$

$$= \frac{256 - 16\cos 3\theta + 16i\sin 3\theta}{257 - 32\cos 3\theta}$$

$\therefore$ real part is $\dfrac{256 - 16\cos 3\theta}{257 - 32\cos 3\theta}$

3 marks Complete solution

2 marks Correct substitution into $S_\infty = \frac{a}{1-r}$ (ie. $S_\infty = \frac{1}{1 - \frac{1}{16}e^{i3\theta}}$)

1 marks Correctly identifies common ratio $r = \frac{1}{16}e^{i3\theta}$

Many students showed no understanding of summing a series, or even that this is a series.
Better solutions worked slowly and methodically, clearly showing each step.

b) ii $T_2 = \frac{1}{16}e^{i3\theta}$

$$= \frac{1}{2^4}(\cos 3\theta + i\sin 3\theta)$$

$$= \frac{1}{2^4}\cos 3\theta + \frac{i}{2^4}\sin 3\theta$$

$T_3 = \frac{1}{256}e^{i6\theta}$

$$= \frac{1}{2^8}(\cos 6\theta + i\sin 6\theta)$$

$$= \frac{1}{2^8}\cos 6\theta + \frac{i}{2^8}\sin 6\theta$$

$\therefore \frac{1}{2^4}\sin 3\theta + \frac{1}{2^8}\sin 6\theta + \dots$ is the imaginary part of the series

$\therefore \frac{1}{2^4}\sin 3\theta + \frac{1}{2^8}\sin 6\theta + \frac{1}{2^{12}}\sin 9\theta + \dots = \dfrac{16\sin 3\theta}{257 - 32\cos 3\theta}$ (from part i)

2 marks Complete solution.

1 mark Clearly shows that $\frac{1}{2^4}\sin 3\theta + \frac{1}{2^8}\sin 6\theta + \dots$ is the imaginary part of the series.

Not well done. Many students need thorough revision of series.

c) i $d > d-1$

$\therefore d^2 > d(d-1)$ (since $d>1$)

$\therefore \frac{1}{d^2} < \frac{1}{d(d-1)}$

Remember that when proving a result, you cannot rely on the truth of that result in your proof.

ii For $d=2$: $\frac{1}{2^2} < \frac{1}{2(2-1)}$ (from part i)

Similarly for $d=3$: $\frac{1}{3^2} < \frac{1}{3(3-1)}$

$\vdots$

$d=n$ $\frac{1}{n^2} < \frac{1}{n(n-1)}$

Summing:

$$\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{(n-1)^2}+\frac{1}{n^2} < \frac{1}{2(2-1)}+\frac{1}{3(3-1)}+\dots+\frac{1}{(n-1)(n-2)}+\frac{1}{n(n-1)}$$

$$< \frac{1}{2-1}-\frac{1}{2}+\frac{1}{3-1}-\frac{1}{3}+\dots+\frac{1}{n-2}-\frac{1}{n-1}+\frac{1}{n-1}+\frac{1}{n} \quad \text{(from part ii)}$$

$$< 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\dots+\frac{1}{n-2}-\frac{1}{n-1}+\frac{1}{n-1}+\frac{1}{n}$$

$$< 1+\frac{1}{n}$$

$$\therefore \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2} < 1+1+\frac{1}{n}$$

$$< 2+\frac{1}{n}$$

$$< 2 \quad \left(\text{as } \lim_{n\to\infty}\frac{1}{n}=0\right)$$

2 marks Complete solution

1 mark Clearly shows $\frac{1}{d^2} < \frac{1}{d-1}-\frac{1}{d}$ for several values of d (eg. $\frac{1}{2^2} < \frac{1}{1}-\frac{1}{2}$ etc)

For a two-mark "show" question, you should expect to show a number of steps.